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The Problem of Irreversibility in Newtonian Dynamics

Michail Zak[1]

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A new type of dissipation function which does not satisfy the Lipschitz condition at equilibrium states is proposed. It is shown that Newtonian dynamics supplemented by this dissipation function becomes irreversible, i.e., it is not invariant with respect to time inversion. Some effects associated with the approaching of equilibria in infinite time are eliminated. New meanings of chaos and turbulence are discussed.

The governing equations of classical dynamics based upon the Newton laws

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}=\frac{\partial L}{\partial q_i}-\frac{\partial R}{\partial \dot{q}_i}, \qquad i=1,2,\ldots,n \tag{1}$$

where L is the Lagrangian, and q, $\dot{q}_i$ are the generalized coordinates and velocities, include a dissipation function $R(\dot{q}_i\dot{q}_j)$ which is associated with the friction forces:

$$F_i(\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_n)=-\frac{\partial R}{\partial \dot{q}_i} \tag{2}$$

The structure of the functions (2) does not follow from Newton's laws, and, strictly speaking, some additional assumptions should be made in order to define it. The "natural" assumption (which has never been challenged) is that these functions can be expanded in Taylor series with respect to an equilibrium state

$$\dot{q}_i=0 \tag{3}$$

[1]Center for Space Microelectronics Technology, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109.

Obviously this requires the existence of the derivatives:

$$\left|\frac{\partial F_i}{\partial \dot{q}_j}\right| < \infty \qquad \text{at} \quad \dot{q}_j \to 0 \tag{4}$$

i.e., F_i must satisfy the Lipschitz condition. This condition allows one to describe the Newtonian dynamics within the mathematical framework of the classical theory of differential equations. However, there is a certain price paid for such a mathematical "convenience": the Newtonian dynamics with dissipative forces (4) remains fully reversible in the sense that the time-backward motion can be obtained from the governing equations by time inversion, $t \to -t$. As stressed by Prigogine (1980), in this view future and past play the same role: nothing can appear in the future which could not already exist in the past, since the trajectories followed by particles can never cross (unless $t \to \pm\infty$). This means that classical dynamics cannot explain the emergence of new dynamical patterns in nature in the same way in which nonequilibrium thermodynamics does.

In order to trivialize the mathematical part of our argumentation, let us consider a one-dimensional motion of a particle decelerated by a friction force:

$$m\dot{v} = F(v) \tag{5}$$

in which m is mass and v is velocity. Invoking the assumption (4), one can linearize the force F with respect to the equilibrium $v = 0$:

$$F \to -\alpha v \qquad \text{at} \quad v \to 0, \quad \alpha = -\left(\frac{\partial F}{\partial v}\right)_{v=0} > 0 \tag{6}$$

and the solution to (5) for $v \to 0$ is

$$v = v_0\, e^{-(\alpha/m)t} \to 0 \qquad \text{at} \quad t \to \infty, \quad v_0 = v(0) \tag{7}$$

As follows from (7), the equilibrium $v = 0$ cannot be approached in finite time. The usual explanation of such an effect is that, to the accuracy of our limited scale of observation, the particle "actually" approaches the equilibrium in finite time. In other words, eventually the trajectories (7) and $v = 0$ become so close that we cannot distinguish them. The same type of explanation is used for the emergence of chaos: if two trajectories originally are "very close" and then they diverge exponentially, the same initial conditions can be applied to either of them, and therefore, the motion cannot be traced.

Hence, there are variety of phenomena whose explanations cannot be based directly upon classical dynamics: in addition, they require some "words" about a scale of observation, "very close" trajectories, etc.

In this note we propose a new structure of the dissipation forces which eliminates the effects discussed above and makes the Newtonian dynamics irreversible. The main properties of the new structure are based upon a violation of the Lipschitz condition (4). Turning to the example (5), let us assume that

$$F=-\alpha v-\alpha_1 v^k, \qquad \alpha_1 \ll \alpha, \qquad k=\frac{p}{p+2}<1, \qquad p \gg 1 \tag{8}$$

in which p is an odd number.

By selecting large p, one can make k close to 1, so that equations (6) and (8) will be almost identical everywhere excluding a small neighborhood of the equilibrium point $v=0$, while, as follows from (8), at this point

$$\left|\frac{\partial F}{\partial v}\right|=(\alpha+k\alpha_1 v^{k-1}) \to \infty \quad \text{at} \quad v \to 0, \qquad \text{i.e.,} \quad F \to -\alpha_1 v^k \quad \text{at} \quad v \to 0 \tag{9}$$

Hence, the condition (4) is violated, the friction force grows sharply at the equilibrium point, and then it gradually approaches the straight line (6). This effect can be interpreted as a jump from static to kinetic friction.

It appears that this "small" difference between the friction forces (6) and (8) leads to fundamental changes in Newtonian dynamics.

First, the time of approaching the equilibrium $v=0$ becomes finite. Indeed, as follows from equations (5) and (9),

$$t_0=-\int_{v_0}^{0} \frac{m\, dv}{\alpha_1 v^k}=\frac{m v_0^{1-k}}{\alpha_1(1-k)}<\infty \tag{10}$$

Obviously this integral diverges in the classical case.

Second, the motion described by equations (5) and (8) has a singular solution $v \equiv 0$ and a regular solution,

$$v=\left[v_0^{1-k}-\frac{\alpha_1}{m}(1-k)t\right]^{1/(1-k)} \tag{11}$$

In a finite time the motion can reach the equilibrium and switch to the singular solution, and this switch is irreversible. It is interesting to note that the time-backward motion

$$v_-=\left\{\left[v_0^{1-k}-\frac{\alpha_1}{m}(1-k)(-t)\right]^{p+2}\right\}^{1/2} \tag{12}$$

is imaginary [one can verify that the classical version of this motion (7) is fully reversible if $t<\infty$].

As shown by Zak (1988, 1989), the equilibrium point $v=0$ of equation (8) represents a terminal attractor which is "infinitely" stable and is intersected by all the attracted transients. Therefore, the uniqueness of the solution at $v=0$ is violated, and the motion for $t<t_0$ [see equation (10)] is totally "forgotten." [This is a mathematical implication of the irreversibility of the dynamics (8).]

So far we have been concerned with the stabilizing effects of dissipative forces. However, as is well known from the dynamics of nonconservative systems, these forces can destabilize the motion when they feed external energy into the system [the transmission of energy from laminar to turbulent flow in fluid dynamics (Drazin, 1984) or from rotations to oscillations in dynamics of flexible systems (Robertson, 1932)]. In order to capture the fundamental properties of these effects in the case of a "terminal" dissipative force (8) by using the simplest mathematical model, let us turn to equation (5) and assume that now the friction force feeds energy into the system:

$$m\dot{v}=\alpha_1 v^k, \qquad k=\frac{p}{p+2}<1, \qquad v\to 0 \tag{13}$$

One can verify that for equation (13) the equilibrium point $v=0$ becomes a terminal repeller (Zak, 1989), and since

$$\frac{d\dot{v}}{dv}=\frac{k\alpha_1}{m}v^{k-1}\to\infty \qquad \text{at} \quad v\to 0 \tag{14}$$

it is "infinitely" unstable. If the initial condition is infinitely close to this repeller, the transient solution will escape it during a finite time period:

$$t_0=\int_{\varepsilon\to 0}^{v_0}\frac{m\,dv}{\alpha_1 v^k}=\frac{m v_0^{1-k}}{\alpha_1(1-k)}<\infty \tag{15}$$

while for a regular repeller, the time would be infinite.

As in the case of a terminal attractor, here the motion is also irreversible: the solution

$$v=\pm\left[\frac{\alpha_1}{m}(1-k)t\right]^{1(1-k)} \tag{16}$$

and the solution (11) are always separated by the singular solution $v\equiv 0$, and each of them cannot be obtained from another by time inversion: the trajectories of attraction and repulsion never coincide.

But in addition to that, terminal repellers possess even more surprising characteristics: the solution (16) becomes totally unpredictable. Indeed, two different motions described by the solution (16) are possible for "almost the same" ($v_0=+\varepsilon\to 0$, or $v_0=-\varepsilon\to 0$ at $t=\to 0$) initial conditions. The most essential property of this result is that the divergence of these two solutions is characterized by an unbounded terminal Lyapunov exponent:

$$\sigma = \lim_{t \to t_0} \left(\frac{1}{t} \ln \frac{t^{1/(1-k)}}{|v_0|} \right) \to \infty \qquad \text{at} \quad |v_0| \to 0 \tag{17}$$

In contrast to the classical case where $t_0 \to \infty$, here σ can be defined in an arbitrarily small time interval t_0, since during this interval the initial infinitesimal distance between the solutions becomes finite. Thus, a terminal repeller represents a vanishingly short, but infinitely powerful "pulse of unpredictability" which is pumped into the system via terminal dissipative forces. Obviously, failure of the uniqueness of the solution here results from the violation of the Lipschitz condition (4) at $v=0$.

As is known from classical dynamics, the combination of stabilizing and destabilizing effects can lead to a new phenomenon: chaos. In order to describe similar effects in dynamics with terminal dissipative forces, let us slightly modify equation (13):

$$m\dot{v} = \alpha_1 v^k \cos \omega t \tag{18}$$

Here stabilization and destabilization effects alternate. With the initial condition $v \to 0$ at $t \to 0$ the exact solution to equation (18) consists of a regular solution

$$v = \pm \left[\frac{\alpha_1(1-k)}{m\omega} \sin \omega t \right]^{1/(1-k)}, \qquad v \neq 0 \tag{19}$$

and a singular solution

$$v = 0 \tag{20}$$

During the first period, $0 < t < \pi/2\omega$, the equilibrium point (20) is a terminal repeller. Therefore, within this interval, the motion can follow one of two possible trajectories (19) (each with the probability $1/2$) which diverge with unbounded Lyapunov exponent (17) at $v=0$. During the next period, $\pi/2\omega < t < 3\pi/2\omega$, the equilibrium point (20) becomes a terminal attractor; the solution approaches it at $t = \pi\omega$ and it remains motionless until $t > 3\pi/2\omega$. After that the terminal attractor converts into a terminal repeller, and the solution escapes again, etc.

It is important to notice that each time the system escapes the terminal repeller, the solution splits into two symmetric branches, so that there are 2^n possible scenarios of the oscillations with respect to the center $v=0$, while each scenario has the probability 2^{-n} (n is the number of cycles). Hence, the motion (19) resembles chaotic oscillations known from classical dynamics: it combines random characteristics with the attraction to a center. However, in the classical case the chaos is caused by a supersensitivity to the initial conditions, while the uniqueness of the solution for fixed initial conditions is guaranteed. In contrast to that, the chaos in the oscillations (19) is caused by the failure of the uniqueness of the solution at the equilibrium points, and it has a well-organized probabilistic structure. Since the time of approaching the equilibrium point $v=0$ by the solution (19) is finite, this type of chaos can be called terminal (Zak, 1991).

Let us turn now to the general case, i.e., to the governing equations (1) and (2), and introduce the following dissipation function:

$$R=\frac{1}{k+1}\sum_i \alpha_i \left|\sum \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j\right|^{k+1} \tag{21}$$

in which $\mathbf{r}_i$ is the radius vector of the ith point of the system. One can verify that the classical case corresponds to $k=1$. As in classical dynamics, this function expresses the dissipation rate of the total energy E:

$$\frac{dE}{dt}=-\sum_i \dot{q}_i\frac{\partial R}{\partial \dot{q}_i}=-(k+1)R \tag{22}$$

Within a small neighborhood of an equilibrium state (where the potential energy can be set zero) the energy E and the dissipation function R have the order, respectively,

$$E \sim \dot{q}_i^2, \qquad R \sim \dot{q}_i^{k+1} \qquad \text{at} \quad E \to 0 \tag{23}$$

Hence, the asymptotic form of equation (22) can be represented as

$$\frac{dE}{dt}=AE^{(k+1)/2} \qquad \text{at} \quad E\to 0, \quad A=\text{const} \tag{24}$$

Obviously, equation (24) is equivalent to equation (5) expressed in terms of energy. This means that all the new properties introduced above are preserved in the general case of Newtonian dynamics with terminal dissipation function (21), i.e., when $k=p/(p+2)<1$. Indeed, since

$$\left|\frac{d\dot{E}}{dE}\right| \to \infty \qquad \text{at} \quad E\to 0 \quad \text{for} \quad k<1 \tag{25}$$

the equilibrium states are represented by terminal attractors or repellers, and therefore the dynamics becomes irreversible. Within the framework of this terminal dynamics, formations of new patterns of motion can be understood as chains of terminal attractions and repulsions: as shown above, during each terminal repulsion the solution splits into two symmetric branches, and the motion can follow each of them with equal probability. Such a scenario can be represented by terminal chaos, which has an exact mathematical formulation and does not depend upon the accuracy to which the initial conditions are known. Driven by nonuniqueness of the solutions at terminal repellers, terminal chaos and consequently the process of emerging of new patterns of dynamical motions possess well-organized probabilistic structures.

For an illustration of the theory, let us turn to fluid dynamics. One of its central problems is to explain how a motion which is described by fully deterministic Navier–Stokes equations can be random. Starting with the simplest shear flow

$$\rho \frac{\partial v_x}{\partial t} = \frac{\partial \sigma_{xy}}{\partial y} \tag{26}$$

where x, y are Cartesian coordinates, ρ, v_x, and σ_{xy} are density, velocity in the x direction, and shear stress, respectively, we will introduce as an analog to equation (9) the following constitutive law:

$$\sigma_{xy} = \mu_1 \left(\frac{\partial v_x}{\partial y}\right)^k \quad \text{at} \quad \frac{\partial v_x}{\partial y} \to 0 \tag{27}$$

which coincides with Newton's formula for $k = 1$.

Combining equations (26) and (27), one obtains a terminal analog of the diffusion equation:

$$\frac{\partial v_x}{\partial t} = k\nu_1 \left(\frac{\partial v_x}{\partial y}\right)^{k-1} \frac{\partial^2 v_x}{\partial y^2}, \qquad \nu_1 = \frac{\mu_1}{\rho}, \qquad \frac{\partial v_x}{\partial y} \to 0 \tag{28}$$

First we will show that equation (28) has a random solution, and then we will discuss its physical interpretation.

Assuming that $v_x(t, y) = v_1(t)v_2(y)$ and separating the variables, one arrives at the following equations:

$$\dot{v}_1 = \lambda v_1^k, \qquad v_2''(v_2')^{k-1} = \frac{\lambda}{\nu_1 k} v_2, \qquad \lambda = \text{const} \tag{29}$$

It can be verified (by substitution) that

$$v_2 = \gamma(y + c)^{(k+1)/(k-1)}, \qquad \gamma = \left[\frac{\lambda (k-1)^{k+1}}{2\nu_1 k(k+1)^k}\right]^{1/(k-1)}, \qquad \lambda > 0,$$
$$c = \left(\frac{v_0}{\gamma}\right)^{(k-1)/(k+1)}, \qquad v_0 = v_2(0) \cdots \tag{30}$$

while for v_1 the solution is similar to (16):

$$v_1 = \pm[\lambda(1-k)t]^{1/(1-k)} \tag{31}$$

and therefore

$$v_x = \pm\gamma[\lambda(1-k)t]^{1/(1-k)}\left[y + \left(\frac{v_0}{\gamma}\right)^{(k-1)/(k+1)}\right]^{(k+1)/(k-1)} \tag{32}$$

It follows from equation (32) that $v_x = 0$ at $t = 0$. But because of the "infinite" instability of the solution (31), any infinitesimal disturbance having the same form as (30) will become finite during a finite time interval. Moreover, since equation (32) has two symmetric branches, with the equal probability 0.5, the solution (30) can be positive or negative. Therefore, equation (32) produces random solutions without any finite random input.

In support of our formal mathematical analysis let us discuss a physical interpretation of the phenomena. The classical version of equation (28) describes the velocity field induced by a sudden move of an infinite plane boundary. But if this boundary has a finite length, it should be replaced by equations of the boundary layer in a flat plate which [with the constitutive law (27)] read

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = k\nu_1\left(\frac{\partial v_x}{\partial y}\right)^{k-1}\frac{\partial^2 v_x}{\partial y^2}, \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{33}$$

Suppose that one considers these equations within an infinitesimal neighborhood of the plate leading edge $y = 0$, where

$$v_x, \quad v_y, \quad \frac{\partial v_x}{\partial x}, \quad \frac{\partial v_x}{\partial y} \to 0 \qquad \text{at} \quad x, y \to 0 \tag{34}$$

In the classical case ($k = 1$) the conditions (34) would lead to zero acceleration of the fluid at the leading edge:

$$\frac{\partial v_x}{\partial t} = 0 \qquad \text{at} \quad x, y \to 0 \tag{35}$$

which is in contradiction to the sudden relative motion between the plate and the fluid. However, for $k < 1$, equations (33) [with the conditions (34)] reduce to equation (28) and have the solution (32). This solution describes the behavior of a fluid characterized by an "infinite" viscosity at the equilibrium state [see equation (27)] which can be associated with a static dry friction. That is why a sudden motion of a plate does not lead to an immediate concentrated jump of velocity gradients: instead it causes a smooth velocity distribution around the leading edge of the plate. Driven by the instability, an infinitesimal velocity distribution (30) becomes finite during a finite time interval.

Obviously the behavior of the solution to equations (33) beyond the infinitesimal neighborhood of the leading edge is described by the classical boundary layer equations. For supercritical Reynolds numbers this solution becomes unstable and it amplifies the contributions (32) coming from the leading edge of the plate. Thus, the combination of the classical mechanism of instability (due to inertia effects) and terminal instability at the fluid equilibrium leads to the probabilistic solutions describing turbulent motions.

In order to illustrate terminal attraction, let us consider a plane incompressible flow with a stream function ψ and the constitutive law:

$$\sigma_{xy}=\mu_1\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)^{k+1}, \qquad v_x=\frac{\partial\psi}{\partial y}, \qquad v_y=-\frac{\partial\psi}{\partial x}, \qquad k<1 \tag{36}$$

Based upon the relationship between the rate of change of the kinetic energy and the dissipation function (Landau, 1953), one obtains

$$\frac{\rho}{2}\frac{\partial}{\partial t}\int_V\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]dx\,dy=-\mu_1\int_V\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)^{k+1}dx\,dy \tag{37}$$

where V is the volume occupied by the fluid.

Suppose that $\psi(t, x, y)$ can be represented as a product $\psi=\tilde{\psi}(t)\bar{\psi}(x, y)$. Then equation (37) reduces to the ordinary differential equation with respect to $\varphi(t)=\tilde{\psi}^2(t)$:

$$\varphi=-\gamma\nu_1\varphi^k \tag{38}$$

and

$$\gamma=\frac{\int_V(\partial^2\bar{\psi}/\partial y^2-\partial^2\bar{\psi}/\partial x^2)^{k+1}\,dx\,dy}{\int_V[(\partial\bar{\psi}/\partial x)^2+(\partial\bar{\psi}/\partial y)^2]\,dx\,dy}=\text{const}, \qquad \nu_1=\frac{\mu_1}{\rho}$$

Equation (38) describes the damping of the fluid motion due to viscons stress (36). The equilibrium state represents a terminal attractor which is approached in a finite time:

$$t_0=\frac{\varphi_0^{1-k}}{\gamma\nu_1(1-k)}, \qquad \varphi_0=\varphi(0) \tag{39}$$

Equation (39) allows one to evaluate k and ν_1 from experimental measurements of t_0.

In conclusion, it should be stressed again that all the new effects of terminal dynamics emerge within vanishingly small neighborhoods of equilibrium states which are the only domains where the governing equations are different from classical.

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